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LIMITED CONTROL OF A RHEONOMOUS MECHANICAL SYSTEM UNDER UNCERTAINTY†

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A rheonomous mechanical system is considered. The kinetic energy of this system is represented in the form of a complete quadratic polynomial with coefficients which depend explicitly on time. It is assumed that the coefficients of this polynomial are unknown and that uncontrolled limited perturbation act on the system. A control law is proposed which enables the system to be brought to a specified final state after a finite time using a force of finite modulus. Piecewise-linear feedbacks are used in the proposed algorithm and the amplification factors of these feedbacks are increased as the system approaches the final state. The algorithm is validated using the second Lyapunov method. The results of numerical modelling are presented. © 2002 Elsevier Science Ltd. All rights reserved.

In the majority of publications concerned with constructing a control for mechanical systems with unknown parameters, either algorithms are proposed which solely ensure the asymptotic stability of the motion of the system, that is, which bring the system to a specified state after a finite time [1, 2], or no constraints whatsoever are imposed on the controls. In practice, as a rule, there are such constraints.

In the case of scleronomous mechanical systems, an approach based on decomposition has been proposed in [3, 4] which enables one, using a finite force, to bring a system, subjected to uncontrolled perturbations and with known inertia matrix, to a specified state after a finite time. Other approaches, which propose the use of programmed trajectories [5, 6] and linear feedbacks with piecewise-constant coefficients [7, 8], have been developed for controlling a Lagrangian system with an unknown kinetic energy matrix.

1. FORMULATION OF THE PROBLEM

A rheonomous mechanical system is considered, the kinetic energy of which is represented in the form of a complete quadratic polynomial with coefficients that depend explicitly on time

$$T(t,q,\dot{q}) = \frac{1}{2}\langle A(t,q)\dot{q},\dot{q}\rangle + \langle a(t,q),\dot{q}\rangle + a_0(t,q) \tag{1.1}$$

Here, $q \in R^n$ is the vector of the generalized coordinates of the system, $\dot{q}$ is the generalized velocity vector, and $\langle\cdot,\cdot\rangle$ denotes a scalar product. It is assumed that the positive-definite symmetric matrix $A(t,q) \in C^1$ is unknown, that its eigenvalues for any t and q belong to the interval $[m, M]$, $0 < m \leqslant M$ and that the partial derivatives are uniformly bounded with respect to the norm, that is,

$$\begin{gathered} mz^2 \leqslant \langle A(t,q)z,z\rangle \leqslant Mz^2, \quad \forall z \in R^n \\ \left\|\frac{\partial A}{\partial q_i}\right\| \leqslant D_1, \quad \left\|\frac{\partial A}{\partial t}\right\| \leqslant D_2, \quad i=1,\ldots,n, \quad D_1, D_2 > 0 \end{gathered} \tag{1.2}$$

The vector function $a(t,q) \in C^1$ and the function $a_0(t,q) \in C^1$ are also assumed to be unknown and to satisfy the conditions

$$\left\|\left(\frac{\partial a}{\partial q}\right)^{\mathrm{T}} - \frac{\partial a}{\partial q}\right\| \leqslant D_3, \quad \left|\frac{\partial a_0}{\partial q} - \frac{\partial a}{\partial t}\right| \leqslant D_4, \quad D_3, D_4 > 0 \tag{1.3}$$

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Above and henceforth, $|\cdot|$ is the Euclidean norm of a vector and $\|\cdot\|$ is the norm of a matrix, which is understood as the norm of the corresponding operator in Euclidean space.

The dynamics of the system being considered are described by Lagrange's equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = S + u \tag{1.4}$$

It is assumed that the system is directly controlled with respect to each degree of freedom, the constraint

$$|u| \leqslant U, \quad U > 0 \tag{1.5}$$

is imposed on the n-dimensional control forces vector, and the generalized forces S are unknown and satisfy the condition

$$|S| \leqslant S_0, \quad S_0 > 0 \tag{1.6}$$

Other known forces can act on the system together with the unknown forces S and the control forces u. We shall assume that the control resources are sufficiently large to compensate these forces and that the quantity U is the maximum possible strength of the control which remains after such compensation.

Problem. Suppose the constants m, M, U and D_j ($j = 1, \ldots, 4$) are given. It is required to construct a control which satisfies constraint (1.5) and indicates the domain of permissible initial states from which system (1.4), under the action of this control, reaches the specified final state of rest $(\bar{q}, 0)$ after a finite time, whatever the matrix A, the vector a, the function a_0 and the perturbations S, which satisfy conditions (1.2), (1.3) and (1.6), may be. Here, it is assumed that the phase variables q, $\dot{q}$ are accessible to measurement at each instant of time. In the case of a scleronomous mechanical system, that is, for a system with a kinetic energy which does not depend explicitly on time and has the form $T(q, \dot{q}) = \langle A(q)\dot{q}, \dot{q}\rangle/2$, a control algorithm, based on the methods of the theory of stability of motion, has been developed in [7, 8] which permits system (1.4) to be brought from an arbitrary initial state to a specified first state of rest after a finite time. Linear feedbacks with piecewise-constant coefficients are used in this algorithm. The coefficients increase and tend to infinity as the trajectory of the system approaches the final state. However, the control forces remain bounded and satisfy the conditions imposed on them. Below, this approach is extended to rheonomous mechanical systems.

2. DESCRIPTION OF THE ALGORITHM

Without loss of generality, we shall assume that the final state coincides with the origin of the coordinates, that is, $\bar{q} = 0$ (this can be achieved using an appropriate change of variables).

We will construct the control in the form of a linear feedback with respect to the generalized coordinates and velocities

$$u = -\alpha_k \dot{q} - \beta_k q, \quad \alpha_k, \beta_k > 0 \tag{2.1}$$

with amplification factors in the form of piecewise-constant functions. We will now describe the algorithm for changing these coefficients.

We will denote the initial state of the systems by $q_0 = q(0)$, $\dot{q}_0 = \dot{q}(0)$ and introduce the function

$$W(q, \dot{q}) = M\dot{q}^2 + (M^2\dot{q}^4 + U^2q^2)^{1/2} \tag{2.2}$$

The quantity $W(q, \dot{q})$ has the dimension of energy and characterizes the remoteness of a point $(q, \dot{q})$ from the final point (0, 0): the set of the level $W(q, \dot{q}) = C$ of the function W in the phase space $q, \dot{q} \in R^{2n}$ is the ellipsoid $2CM\dot{q}^2 + U^2q^2 = C^2$, which contracts to the origin of the coordinates (0, 0) when $C \to 0$.

We put

$$W_0 = W(q_0, \dot{q}_0), \quad W_k = \frac{W_0}{2^k}, \quad k = 1, 2, \ldots \tag{2.3}$$

The sets of the level of the function $W(q, \dot{q})$ corresponding to the constants W_k are a family of ellipsoids which contract to zero as k increases. We will denote the instant if time when the trajectory hits the ellipsoid $W(q, \dot{q}) = W_1$ for the first time by t_1 and we put $q_1 = q(t_1)$, $\dot{q}_1 = \dot{q}(t_1)$. It will be shown below that, in the case of the chosen control algorithm, the trajectory of the system tends to the origin of coordinates and therefore such an instant of time exists. The instant when the trajectory of the system hits the ellipsoid $W(q, \dot{q}) = W_2$ for the first time is denoted by t_2. We put $q_2 = q(t_2)$, $\dot{q}_2 = \dot{q}(t_2)$ and so on.

The sequence $\{t_k\}$ determines the instants when the coefficients α_k, β_k in Eq. (2.1) are changed. We will specify the values of these coefficients in the half-interval of time $[t_k, t_{k+1})$ $(k = 0, 1, \ldots)$ as follows:

$$\beta_k = \frac{U^2}{2W_k}, \qquad \alpha_k^2 = m\beta_k \tag{2.4}$$

In the phase space R^{2n}, the trajectory of the motion of the mechanical system consists of segments of the trajectories of different systems of differential equations: the k-th segment joins the points $(q_k, \dot{q}_k)$ and $(q_{k+1}, \dot{q}_{k+1})$ and corresponds to a system of the form (1.4), (2.1) in which the amplification factors α_k, β_k are constant and are determined by formula (2.4). All the points $(q_k, \dot{q}_k)$ lie on the corresponding ellipsoids $W(q, \dot{q}) = W_k (k = 0, 1, \ldots)$ (Fig. 1).

Note that, generally speaking, the function W is not a monotonically decreasing function along the trajectory of the system, despite the fact that the trajectory tends to the origin of coordinates (0, 0). Hence, a trajectory can have more than one point of intersection with certain ellipsoids. We will assume, for example, that, after designating the new coefficients at the instant t_k, the trajectory of the system has started to "move away" from the origin of coordinates (0, 0) and again intersected the ellipsoid $W(q, \dot{q}) = W_{k+1}$ at the instant $t' > t_k$. The index k and the coefficients α_k, β_k do not change at the instant t'.

Hence, when implementing the algorithm, it is sufficient to measure the actual values of the phase variables of the system q, $\dot{q}$ and to store in the memory the actual value of the index k, which is equal to the number of the smallest ellipsoid which the trajectory of the system has already visited. Since, in expression (2.2) for the function W, only the known parameters of the problem appear, apart from the phase variables, the value of the function $W(q(t), \dot{q}(t))$ can be calculated at any instant. Each time the value of W decreases by a factor of two, the index k increases by unity, the coefficient α increases by a factor of $\sqrt{2}$ and the coefficient β increases by a factor of 2.

3. VALIDATION OF THE ALGORITHM

We will use Lyapunov's second method to validate the algorithm. Consider the kth segment of the trajectory for a certain fixed $k \geqslant 0$. This segment begins at the point $(q_k, \dot{q}_k)$ at the instant t_k and corresponds to system (1.4), (2.1) with constant feedback coefficients specified by formulae (2.4). We will now show that there is such an instant t_{k+1} when the trajectory of the system hits the ellipsoid $W(q, \dot{q}) = W_{k+1}$.

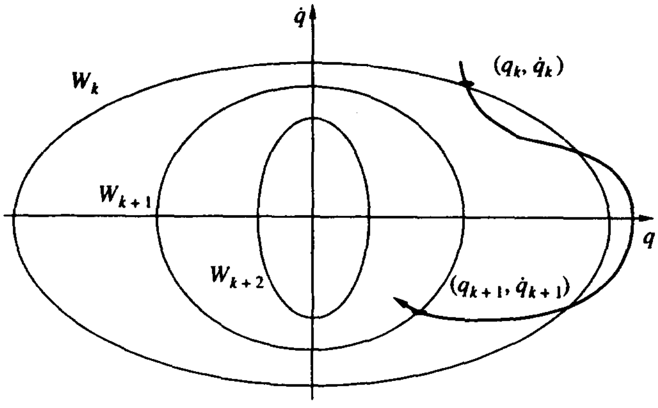

Fig. 1

The Lyapunov function and its evaluation. We put

$$\varepsilon_k = \sqrt{m\beta_k}/(4M) \tag{3.1}$$

and introduce Lyapunov's function

$$V^k(t,q,\dot q) = \frac{1}{2}\langle A(t,q)\dot q,\dot q\rangle + \frac{1}{2}\beta_k q^2 + \varepsilon_k\langle A(t,q)\dot q, q\rangle \tag{3.2}$$

The expression for the function V^k contains the kinetic energy matrix $A(t, q)$, which is assumed to be unknown. We estimate the value of this function at an arbitrary point $(t, q, \dot q)$ of the expanded phase space in terms of known functions. The relations

$$|\varepsilon_k\langle A\dot q, q\rangle| \leqslant \frac{1}{8}\langle A\dot q,\dot q\rangle + 2\varepsilon_k^2\langle Aq,q\rangle \leqslant \frac{1}{8}\left(\langle A\dot q,\dot q\rangle + \frac{m\beta_k}{M^2}\langle Aq,q\rangle\right)$$

hold by virtue of the Cauchy inequality and expression (3.1), and it follows from condition (1.2) that

$$\frac{m\beta_k}{M^2}\langle Aq,q\rangle \leqslant \frac{m\beta_k}{M}q^2 \leqslant \beta_k q^2$$

Substituting the inequalities obtained into relation (3.2) and again using condition (1.2), we obtain the following limits for the function V^k

$$V_-^k(q,\dot q) \leqslant V^k(t,q,\dot q) \leqslant V_+^k(q,\dot q) \tag{3.3}$$

where

$$V_-^k(q,\dot q) = \frac{3}{8}(m\dot q^2 + \beta_k q^2), \qquad V_+^k(q,\dot q) = \frac{5}{8}(M\dot q^2 + \beta_k q^2) \tag{3.4}$$

We will now establish some relations connecting the functions $V_+^k(q, \dot q)$ and $W(q, \dot q)$. Substituting the formula for the coefficient β_k from (2.4) into expression (3.4), we obtain for the function V_+^k.

$$V_+^k(q_k,\dot q_k) = \frac{10M\dot q_k^2 W_k + 5U^2 q_k^2}{16W_k} \tag{3.5}$$

By construction, the point $(q_k, \dot q_k)$ lies on the ellipsoid with number k. It follows from this and from definition (2.2) of the function W that

$$W_k = W(q_k,\dot q_k) = M\dot q_k^2 + (M^2\dot q_k^4 + U^2 q_k^2)^{1/2}$$

Using this equality, the numerator in expression (3.5) is reduced to the form $5W_k^2$ and the relation

$$V_+^k(q_k,\dot q_k) = 5W_k/16 \tag{3.6}$$

which connects the functions $V_+^k(q, \dot q)$ and $W(q, \dot q)$, follows from this. This means that, for any k, the ellipsoid with number k is the level set of the quadratic form $V_+^k(q, \dot q)$, which corresponds to the value $5W_k/16$. In accordance with the algorithm, the point $(q(t), \dot q(t))$, when $t \in [t_k, t_{k+1})$, lies outside the $(k + 1)$-th ellipsoid, that is, outside the level set $V_+^{k+1}(q, \dot q) = 5W_{k+1}/16$, and, therefore,

$$V_+^{k+1}(q(t),\dot q(t)) > 5W_{k+1}/16 = 5W_k/32, \quad t_k \leqslant t < t_{k+1}$$

The equality $\beta_{k+1} = 2\beta_k$ holds by virtue of formulae (2.3) and (2.4), and the relation

$$M\dot q^2 + \beta_k q^2 \geqslant (M\dot q^2 + \beta_{k+1}q^2)/2$$

follows from this. Consequently, the limit

$$V_+^k(q(t),\dot{q}(t)) \geqslant V_+^{k+1}(q(t),\dot{q}(t))/2 \geqslant 5W_k/64 \tag{3.7}$$

holds in the kth segment of the trajectory.

Negative definiteness of the derivative of Lyapunov's function. We will now calculate the derivative $\dot{V}^k$. We introduce the notation.

$$B(t,q)=\left(\frac{\partial a}{\partial q}(t,q)\right)^{\mathrm{T}}-\frac{\partial a}{\partial q}(t,q),\qquad b(t,q)=\frac{\partial a_0}{\partial q}(t,q)-\frac{\partial a}{\partial t}(t,q) \tag{3.8}$$

and differentiate the function V^k according to (1.4) and (2.1). We obtain

$$\dot{V}^k(t,q,\dot{q})=-\left\langle\left[\alpha_k I-\varepsilon_k A+\frac{1}{2}\frac{\partial A}{\partial t}-\frac{\varepsilon_k}{2}\sum_{i=1}^n q_i\frac{\partial A}{\partial q_i}\right]\dot{q},\dot{q}\right\rangle-$$

$$-\varepsilon_k\beta_k q^2-\varepsilon_k\alpha_k\langle\dot{q},q\rangle+\langle S+b,\dot{q}+\varepsilon_k q\rangle-\varepsilon_k\langle B\dot{q},q\rangle \tag{3.9}$$

where I is the identity

We will now estimate the individual terms in expression (3.9). Using the Cauchy inequality and relations (1.3) and (3.8), we obtain

$$|\varepsilon_k\alpha_k\langle\dot{q},q\rangle|\leqslant\frac{\alpha_k}{4}\dot{q}^2+\varepsilon_k^2\alpha_k q^2,\quad |\varepsilon_k\langle B\dot{q},q\rangle|\leqslant\frac{D_3}{2}\dot{q}^2+\frac{\varepsilon_k^2 D_3}{2}q^2 \tag{3.10}$$

Using the inequality $|2\varepsilon_k\langle\dot{q},q\rangle|\leqslant\dot{q}^2/16+16\varepsilon_k^2q^2$, expression (3.1) for ε_k and relation (3.7), we estimate the quantity $|\dot{q}+\varepsilon_k q|$ as follows

$$(\dot{q}+\varepsilon_k q)^2\leqslant\frac{17}{16}\dot{q}^2+17\varepsilon_k^2q^2\leqslant\frac{17}{16M}(M\dot{q}^2+\beta_k q^2)=\frac{17}{10M}V_+^k(q,\dot{q})=$$

$$=\frac{17}{10MV_+^k(q,\dot{q})}(V_+^K(q,\dot{q}))^2\leqslant\frac{1088}{50MW_k}(V_+^k(q,\dot{q}))^2$$

whence, taking the second expression in (3.4) into account, we obtain

$$|\langle S+b,\dot{q}+\varepsilon_k q\rangle|\leqslant|S+b|\sqrt{\frac{17}{2MW_k}}(M\dot{q}^2+\beta_k q^2) \tag{3.11}$$

The relation

$$\left|\frac{\varepsilon_k}{2}\sum_{i=1}^n q_i\frac{\partial A}{\partial q_i}\right|\leqslant\frac{\sqrt{n}D_1}{2}\varepsilon_k|q| \tag{3.12}$$

holds by virtue of (1.2) and the inequality $\sum_{i=1}^n|q_i|\leqslant\sqrt{n}|q|$.

Substituting inequalities (3.10)–(3.12) into expression (3.9) and making use of conditions (1.2), (1.3) and (1.6), we arrive at the following limit for the derivative of the function V^k along the kth segment of the trajectory

$$\dot{V}^k(t,q,\dot{q})\leqslant-\left(\varepsilon_k\beta_k-\varepsilon_k^2\alpha_k-\beta_k\sqrt{\frac{17}{2MW_k}}(S_0+D_4)-\frac{\varepsilon_k^2D_3}{2}\right)q^2-$$

$$-\left(\frac{3\alpha_k}{4}-\varepsilon_k M-\frac{D_2+D_3}{2}-\sqrt{\frac{17M}{2W_k}}(S_0+D_4)-\frac{\sqrt{n}D_1}{2}\varepsilon_k|q|\right)\dot{q}^2 \tag{3.13}$$

We will now show that, under certain additional assumptions, the derivative $\dot{V}^k$ will be negative-definite. We assume that

$$\Omega = \min\left\{\frac{\sqrt{15}MU}{8\sqrt{n}D_1}, \frac{mU^2}{32D_2^2}, \frac{mU^2}{32D_3^2}\right\}$$

and introduce the sets

$$G = \{(q,\dot{q}) \in R^{2n} : W(q,\dot{q}) \leqslant \Omega\}, \quad G_k = \left\{(q,\dot{q}) : |q| < \sqrt{\frac{5}{3}\frac{W_k}{U}}\right\}, \quad k = 0, 1 \ldots$$

The inequality $3q_k^2U^2 \leqslant 5W^2(q_k, \dot{q}_k)$ results from the definition of the function W, and, by virtue of relations (2.3), it follows from this that the point $(q_k, \dot{q}_k)$ lies in the domain G_k.

Lemma 1. Suppose the initial point $(q_k, \dot{q}_k)$ of the kth segment belongs to the set G, the matrix A, the vector functions S, a and the function a_0 satisfy conditions (1.2), (1.3) and (1.6) and

$$S_0 + D_4 \leqslant \sqrt{\frac{m}{17M}\frac{U}{8}} \tag{3.14}$$

Then on the part of the trajectory which starts at the point $(q_k, \dot{q}_k)$, the derivative of the function V^k lies outside the ellipsoid $W(q, \dot{q}) = W_{k+1}$ and in the set G_k and, by virtue of system (1.4), (2.1), (2.4), satisfies the inequality

$$\dot{V}^k(t,q,\dot{q}) \leqslant -\frac{3\alpha_k}{40M}V^k(t,q,\dot{q}) \tag{3.15}$$

Proof. By the condition of the lemma, $W(q_k, \dot{q}_k) \leqslant \Omega$ and, consequently, $D_2^2, D_3^2 \leqslant mU^2/(32W_k)$. It follows from this and from definitions (2.4) and (3.1) of the numbers ε_k, α_k that

$$\frac{D_2 + D_3}{2} \leqslant \frac{\alpha_k}{4}, \quad \frac{\varepsilon_k^2 D_3}{2} \leqslant \frac{\alpha_k\beta_k}{64M} \tag{3.16}$$

From condition (3.14) and formulae (2.4), we obtain

$$\sqrt{\frac{17M}{2W_k}}(S_0 + D_4) \leqslant \frac{\alpha_k}{8}, \quad \beta_k\sqrt{\frac{17}{2MW_k}}(S_0 + D_4) \leqslant \frac{\alpha_k\beta_k}{8M} \tag{3.17}$$

By virtue of relations (2.4) and (3.1), we have

$$\varepsilon_k M = \frac{\alpha_k}{4}, \quad \varepsilon_k\beta_k - \varepsilon_k^2\alpha_k = \frac{\alpha_k\beta_k}{4M}\left(1 - \frac{m}{4M}\right) \geqslant \frac{3\alpha_k\beta_k}{16M} \tag{3.18}$$

The inequality $D_1 \leqslant \sqrt{15}MU/(8\sqrt{n}Wk)$ follows from the condition $W(q_k, \dot{q}_k) < \Omega$. Since the section of the trajectory being considered lies in the set G_k, we have

$$\varepsilon_k |q| \leqslant \sqrt{5}\alpha_k W_k/(4\sqrt{3}MU)$$

and, consequently,

$$\frac{\sqrt{n}D_1}{2}\varepsilon_k |q| \leqslant \frac{5\alpha_k}{64} \tag{3.19}$$

Substituting inequalities (3.16)–(3.19) into (3.13) and making use of the equalities (3.4), we arrive at the relations

$$\dot{V}^k(t,q,\dot{q}) \leqslant -\frac{3\alpha_k}{64M}(M\dot{q}^2 + \beta_k q^2) \leqslant -\frac{3\alpha_k}{40M}V_+^k(q,\dot{q})$$

whence the assertion of the lemma follows, by virtue of limits (3.3).

Lemma 2. Suppose the matrix A, the vector-functions S, a and the functions a_0 satisfy conditions (1.2), (1.3), (1.6) and (3.4) and that $(q_k, \dot{q}_k) \in G$. Inequality (3.15) is then satisfied in the kth interval of the trajectory.

Proof. It has already been established above that $(q_k, \dot{q}_k) \in G_k$. By virtue of Lemma 1, to prove Lemma 2 it is sufficient to show that the kth section of the trajectory as a whole lies in the domain G_k.

Let us assume the opposite. Suppose t' is the first instant when the trajectory leaves the domain G_k, that is

$$q^2(t') = \frac{5W_k^2}{3U^2} \tag{3.20}$$

On the other hand, it follows from definitions (2.4) and (3.1) of the coefficients ε_k, β_k and from relations (3.3) and (3.4) that

$$\varepsilon_k^2 q^2(t') = \frac{m}{16M^2}\beta_k q^2(t') \leqslant \frac{m}{16M^2}(M\dot{q}^2(t') + \beta_k q^2(t')) =$$
$$= \frac{m}{6M^2} V_-^k(q(t'), \dot{q}(t')) \leqslant \frac{m}{6M^2} V^k(t', q(t'), \dot{q}(t'))$$

Since, when $t_k \leqslant t < t'$, the part of the trajectory being considered lies in the domain G_k, then, by virtue of Lemma 1, the function V_k decreases along it and, using relation (3.2), the limit can be continued as follows:

$$\varepsilon_k^2 q^2(t') < \frac{m}{6M^2} V^k(t_k q(t_k), \dot{q}(t_k)) \leqslant \frac{m}{6M^2} V_+^k(q(t_k), \dot{q}(t_k)) = \frac{5m}{96M^2} W_k$$

Consequently

$$q^2(t') < \frac{5mW_k}{96M^2\varepsilon_k^2} = \frac{5W_k^2}{3U^2}$$

which contradicts condition (3.20).

It follows from the assertions of Lemmas 1 and 2 that, outside the ellipsoid $W(q, \dot{q}) = W_{k+1}$, the function V_k strictly decreases along a trajectory of system (1.4), (2.1), (2.4) and, by virtue of relations (3.3)–(3.6), there is an instant t_{k+1} when the trajectory hits the ellipsoid with number $k + 1$.

It is clear that, if the initial state of the system $(q_0, \dot{q}_0)$ belongs to the set G, the ellipsoid $W(q, \dot{q}) = W_0$ and, together with it, all the remaining ellipsoids $W(q, \dot{q}) = W_k$ $(k = 1, 2, \ldots)$ lie in this set as a whole. Consequently, all the points $(q_k, \dot{q}_k)$ also belong to G and the assertions of Lemmas 1 and 2 are applicable to any of the sections constituting the trajectory of motion of the system.

Estimation of the time of motion. We will now show that the system reaches the origin of coordinates after a finite time. In order to estimate the time of motion along the kth section of the trajectory, we integrate inequality (3.15) and obtain

$$t_{k+1} - t_k \leqslant \frac{40M}{3\alpha_k} \ln \frac{V^k(t_k, q_k, \dot{q}_k)}{V^k(t_{k+1}, q_{k+1}, \dot{q}_{k+1})} \tag{3.21}$$

By virtue of relations (2.4) and (3.3)–(3.7), we have

$$V^k(t_k, q_k, \dot{q}_k) \leqslant \frac{5}{16} W_k$$

$$V^k(t_{k+1}, q_{k+1}, \dot{q}_{k+1}) \geqslant V_-^k(q_{k+1}, \dot{q}_{k+1}) = \frac{3}{8}(m\dot{q}_{k+1}^2 + \beta_k q_{k+1}^2) \geqslant$$
$$\geqslant \frac{3m}{16M}(M\dot{q}_{k+1}^2 + \beta_{k+1} q_{k+1}^2) = \frac{3m}{10M} V_+^{k+1}(q_{k+1}, \dot{q}_{k+1}) = \frac{3m}{64M} W_k$$

Substituting these relations and expression (2.4) for α_k into inequality (3.21), we obtain the following estimate of the time of motion from the point $(q_k \dot{q}_k)$ up to the point $(q_{k+1}, \dot{q}_{k+1})$

$$t_{k+1}-t_k \leqslant \tau 2^{-k/2}, \quad \tau=\frac{40M\sqrt{2W_0}}{3\sqrt{mU}}\ln\frac{20M}{3m}, \quad k=0,1,\ldots$$

The total time of motion of the system up to the final state T_* does not exceed the sum of the series

$$T_* \leqslant \tau\sum_{k=0}^{\infty} 2^{-k/2} = \frac{\tau\sqrt{2}}{\sqrt{2}-1} \tag{3.22}$$

Consequently, the proposed control algorithm brings system (1.4) to the origin of coordinates after a finite time.

We will now verity that condition (1.5) is satisfied along the trajectory of the motion. To do this, we estimate the modulus of the vector of the control forces along the kth section of the trajectory using the Cauchy inequality and relations (2.4), (3.3) and (3.4) as follows:

$$|u|^2 \leqslant 2(\alpha_k^2\dot{q}^2+\beta_k^2q^2)=2\beta_k(m\dot{q}^2+\beta_kq^2)=\frac{16}{3}\beta_kV_-^k(q,\dot{q})\leqslant\frac{16}{3}\beta_kV^k(t,q,\dot{q})$$

Since the function V_k decreases in the half-interval $[t_k, t_{k+1})$ we can use relation (3.6) to continue the estimate as follows:

$$|u|^2 \leqslant \frac{16}{3}\beta_kV^k(t_k,q_k,\dot{q}_k)\leqslant\frac{16}{3}\beta_kV_+^K(q_k,\dot{q}_k)=\frac{5}{3}\beta_kW_k=\frac{5}{6}U^2$$

whence inequality (1.15) follows.

Modification of the algorithm. It follows from the arguments presented above that the system reaches the point (0, 0) after a finite time if the initial stage belongs to the ellipsoid G. Note that any point of the form $(\bar{q}, 0)$ in the phase space of the system can be chosen as the final state. Here, the set of ellipsoids on which a change in the amplification factors occurs is found to be displaced by the vector $\bar{q}$ while the parameters of the ellipsoids remain as before. We will now show that, by making use of this fact and modifying the proposed algorithm, it is possible to extend the set of permissible initial states considerably.

Suppose that

$$(q_0,\dot{q}_0)\in G_*, \quad G_*=\left\{(q,\dot{q})\in R^{2n}:\dot{q}^2\leqslant\frac{\Omega}{2M}\right\} \tag{3.23}$$

We first transfer the system to the point $q = q_0$, $\dot{q} = 0$. To do this, we make the change of variables $q' = q - q_0$. In the new variables $G' = \{(q', \dot{q}')\colon W(q', \dot{q}') \leqslant \Omega\}$, which is analogous to the set G considered earlier, is an ellipsoid with its centre at the point $q' = \dot{q}' - 0$, and the initial state of the system, that is, the point $q_0' = 0$, $\dot{q}_0' = \dot{q}_0$, belongs to this set by virtue of inclusion (3.23) and definition (2.2) of the function W. Consequently, the control law $u = -\alpha_k\dot{q}' - \beta_kq'$ with the above-mentioned algorithm for changing the coefficients α_k, β_k brings the system after a finite time to the centre of this ellipsoid, that is, to the point $q = q_0$, $\dot{q} = 0$.

In the phase space q, $\dot{q}$, we chose a finite sequence of points $(\bar{q}^j, 0), j = 1, 2, \ldots, J$, such that $\bar{q} = q_0$, $\bar{q}^J = 0$ and

$$|\bar{q}^j-\bar{q}^{j-1}|\leqslant\Omega/U, \qquad j=2,\ldots,J \tag{3.24}$$

The transfer of the system from the point $(\bar{q}^1, 0) = (q_0, 0)$ to the point $(\bar{q}^J, 0) = (0, 0)$, that is, to the origin of coordinates, is accomplished after $J - 1$ steps, on applying the control algorithm again each time. The point $(\bar{q}^j, 0)$ corresponds to the initial state of the system at the j-th step and the point $(\bar{q}^{j+1}, 0)$ corresponds to the final state. It follows from inequality (3.24) and definition (2.2) of the function W that, for any j, the point $(\bar{q}^j, 0)$ belongs to the ellipsoid $G^j = \{(q, \dot{q}) : W(q - \bar{q}^{j+1}, \dot{q}) < \Omega$ with its centre at $(\bar{q}^{j+1}, 0)$. This ellipsoid is the set of permissible initial states of the system in order for it to be brought to the final state $(\bar{q}^{j+1}, 0)$ at the jth step. Consequently, the control law $u = -\alpha_k\dot{q} - \beta_k(q - \bar{q}^{j+1})$ with the above-mentioned algorithm for rechanging the coefficients α_k, β_k transfers the system from the point $(\bar{q}^j, 0)$ to the centre of this ellipsoid, that is, to the point $q = \bar{q}^{j+1}$, $\dot{q} = 0$ after a finite time. Hence, after $J - 1$ steps, system (1.4) finds itself in the final state (0, 0).

The basic result and its discussion. The following theorem sums up the arguments which have been presented above.

Theorem. Suppose the matrix A, the vector functions S, a and the function a_0 satisfy conditions (1.2), (1.3), (1.6) and (3.4) and $(q_0, \dot{q}_0) \in G_*$. Then, the proposed control law transfers system (1.4) from the initial state $(q_0, \dot{q}_0)$ to the phase space origin in a finite time. Hence, the control forces will satisfy constraint (1.5).

We will now compare this result with the results obtained earlier in [3–8]. It has already been noted above that the approach used here is an extension of that developed in [7, 8] for scleronomous systems to rheonomous mechanical systems. In the case of scleronomous systems, the set of permissible initial states is identical to the whole of the phase space, that is, the system is brought from an arbitrary initial position to a specified final state. In the case of rheonomous systems, the set of permissible initial states (3.23) is a "band" in the phase space R^{2n}: the condition $\dot{q}_0^2 \leqslant \Omega/(2M)$ is imposed on the initial velocities.

Note that only the known parameters of the problem appear in the definition of the set G_* and the expressions for the function W and the amplification factors α_k, β_k. To implement the algorithm, it is sufficient to know the value of m, M and U and, also, the values of the phase variables of the system at each actual instant of time. The constants D_1, D_2, D_3 only occur in the conditions determining the set of permissible initial states G_*. These conditions, as well as the constraints on the vector function $a(t, q)$, the function $a_0(t, q)$ and the perturbing forces S in relations (3.14), are only sufficient conditions for transferring the system to the final state. The algorithm proposed can therefore also be formally applied in cases when constraints (3.14) are not satisfied and the initial state of the system does not belong to the set G_*. Simulation of the dynamics of different mechanical systems shows that the algorithm is also effective beyond the limits of the sufficient conditions which have been presented.

The set of permissible initial states in the case of the control laws developed earlier in [3–6], which are based on the principle of decomposition, is also the whole of the phase space. However, these laws are only applicable to scleronomous systems.

The conditions imposed on the perturbing force S, obtained earlier in [3, 4], are analogous to condition (3.14) since, in the case of a scleronomous system, one can put $D_4 = 0$. Furthermore, unlike in this paper, it was assumed in [3, 4] that the kinetic energy matrix A is known.

The condition, developed in [5, 6], for transferring the system to the final state using a control law is specified by the inequality $U > S$, that is, simple superiority of the control over the disturbance is sufficient to achieve the aim of the control. This condition suitably distinguishes this approach from the other approaches considered here in which multiple excess of the control resources over the disturbance is required and the relation between them depends on the other parameters of the system.

A more detailed comparison of the control laws considered for scleronomous mechanical systems has been presented in [9]. In particular, the above-mentioned algorithms were used to transfer a mass point of unknown mass, moving along a straight line, to the origin of coordinates. The time optimal control for this problem is known [10]. A comparison shows that, in the case of this simplest mechanical system, the time of motion under the action of the control laws proposed earlier in [3–8] differs from the optimal time by a factor of two or three.

Together with the approaches discussed here for constructing a control for mechanical systems under uncertainty, there is a set of other approaches which ensure the asymptotic stability of a specified state of a system, that is, which ensure that it is transferred to the final position in a finite time. In spite of the fact that, in practical applications, the system is always only brought to within a certain neighbourhood of the specified state, the formulation of the problem of transfer in a finite time makes sense. As the dimensions of the final neighbourhood become smaller, the time of the motion of the system, under the action of the algorithm, which ensures asymptotic stability tends to infinity while the time of the motion of the system under the action of an algorithm which guarantees the finiteness of the process remains finite. Consequently, the last algorithm is more effective from the point of view of its operation speed.

4. RESULTS OF SIMULATION

We will illustrate the operation of the algorithm using numerical simulation of the rotation of a body with a moment of inertia which changes with time. Consider a system consisting of a weightless rod and a point mass of unknown mass m_0 (the right-hand side of the upper part of Fig. 2), which moves along the rod in an uncertain way. It is assumed that the rod rotates in a horizontal plane about one

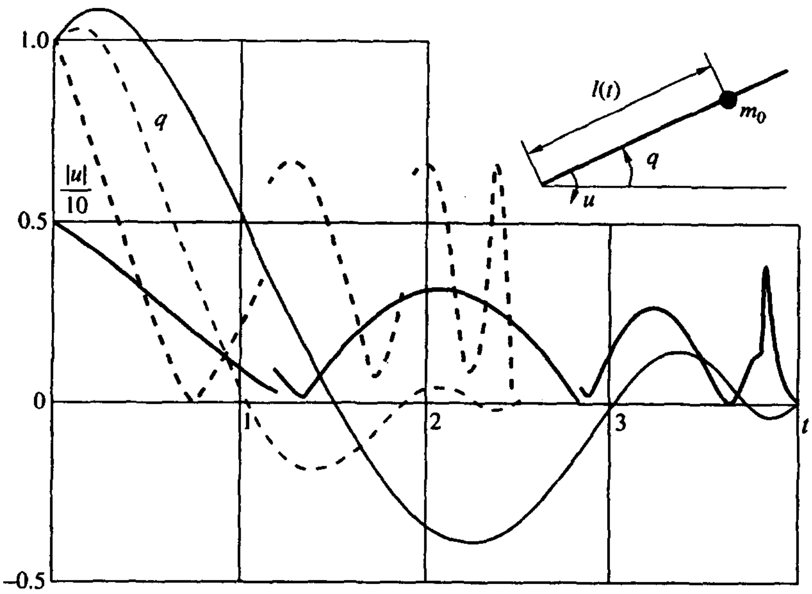

Fig. 2

of its ends under the action of a control moment u. We will denote the angular coordinate and the angular velocity of the rod by q and $\dot{q}$ and the distance from the axis of rotation to the point mass by $l(t)$. In the notation adopted above, the individual terms in expression (1.1) for the kinetic energy of the system take the form

$$A(t) = m_0 l^2(t), \quad a \equiv 0, \quad a_0(t) = m_0 \dot{l}^2(t)/2$$

and the equations of motion take the form

$$m_0 l^2(t)\ddot{q} + 2m_0 l(t)\dot{l}(t)\dot{q} = S + u \tag{4.1}$$

In this case, the moment of the dry friction forces which acts on the rod serves as the unknown generalized force S. In the simulation, the constants m, M and U and the mass m_0, the perturbations S and the law of motion of the point mass along the rod $l(t)$, which are also assumed to be unknown, were taken as follows:

$$m = 0.25\,\text{kg}, \quad M = 2.25\,\text{kg}, \quad U = 10\,\text{N m}$$
$$S = -0.1\,\text{sign}(\dot{q})\text{N m}, \quad m_0 = 1\,\text{kg}, \quad l(t) = 1 + \tfrac{1}{2}\sin\omega t\ \text{m}$$

Using the proposed control law, the rod is shifted from the initial state $q_0 = 1$ radian, $\dot{q}_0 = 1$ radian/s into the final state $q = \dot{q} = 0$. Integration of Eq. (4.1) was discontinued when the Euclidean distance from the actual point of the trajectory to the final point in the phase space $(q, \dot{q}) \in R^2$ became less than 0.01.

The results of the simulation for the case when $\omega = 1$ are shown in Fig. 2. The time-dependence of the angle of rotation of the rod q is represented by the thin solid curve and the time-dependence of the quantity $|u|/10$ is represented by the heavy solid curve (the discontinuous curve). The coefficients α_k and β_k were changed seven times during the course of the integration. The total time of motion was found to be equal to $T_* = 3.98$ s.

It is clear that constraint (1.5) is satisfied with a considerable margin. The motion of the system, which is controlled using the law (2.1), was simulated with feedback coefficients α_k, β_k which were twice those prescribed by algorithm (2.4). The time-dependences of the angle of rotation of the rod (the thin curve) and of the quantity $|u|/10$ (the heavy curve), for the case of this control law, are represented by the dashed curves in Fig. 2. In this case, the time of motion was reduced to $T_* = 2.53$ s and, as previously, the control u satisfies constraint (1.5).

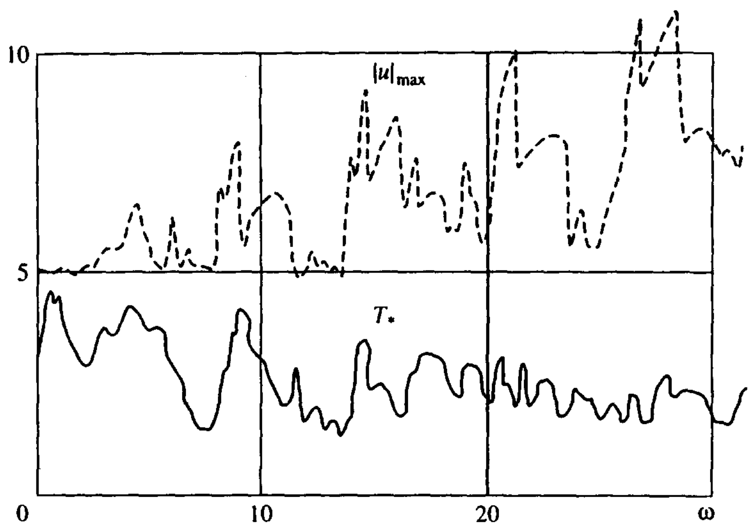

Fig. 3

In order to estimate the efficiency of the algorithm when condition $(q_0, \dot{q}_0) \in G_*$ of the theorem formulated above is violated, the dynamics of system (4.1) were simulated for different values of ω. The dependence of the overall time of motion of the system on ω up to the final state, where $\omega \in [0, 10\pi]$, is represented by the continuous curve in Fig. 3. In this case

$$\dot{A}(t) = m_0\omega\left(1 + \frac{1}{2}\sin\omega t\right)\cos\omega t$$

and the constant D_2 from constrains (1.2) satisfies the inequality $\omega \leqslant D_2$. Consequently, $\Omega < mU^2/(32\omega^2)$ and, in the case of the chosen values of the system parameters, for a large part of the interval $0 \leqslant \omega \leqslant 10\pi$ the initial state $q_0 = 1, \dot{q}_0 = 1$ does not lie in the domain G_*. Nevertheless, the proposed control law does bring the system to the final state.

The dependence of the absolute magnitude of the control moment u, which is realized when the algorithm is applied, on the parameter ω is shown by the dashed curve in Fig. 3. It is clear that constraints (1.5) are satisfied for all the values of $\omega < 26$ considered.

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